

An Exponential Diophantine Equation - One Order Higher than Fermat's Equation

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To Yuri Manin with respect and admiration.

Abstract

We formulate an exponential Diophantine equation, which is in some sense one order higher than Fermat's Last Theorem. We also give three examples of solutions to this exponential Diophantine equation and formulate a conjecture.

Key words: Diophantine equation, exponential equation

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1 An Exponential Diophantine Equation

Fermat Last Theorem was an open conjecture for about 300 years. It was proven not long ago by Andrew Wiles [W].

Theorem 1.1 (*Fermat's Last Theorem; Wiles*) *For an integer $n > 2$ there are no non-trivial integer solutions of*

$$x^n + y^n = z^n.$$

Where by trivial we mean one of the variables x, y, z to be zero.

Earlier proofs, for a particular exponent n or for a large class of exponents n , were based on algebraic number theory. One can look at the book by Ireland and Rosen [IR] for Euler's proof for $n = 3, 4$, and [K] for Kummer's proof for a large class of exponents n . Wiles proof was based on proving Taniyama-Shimura conjecture that all elliptic curves over \mathbb{Q} are modular.

More recently, a slight modification of the formulation of the Fermat Last Theorem was related to Hilbert modular surfaces [BDDDDV]. One may consider the following Diophantine equation:

$$x^k + y^m = z^n, \tag{1.1}$$

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where k, m and n could be different positive integers, (see [Kr]).

We generalize Equation (1.1) so that the operation are in some sense one order of complexity higher. Instead of addition we consider multiplication and instead of multiplication we consider exponentiation. Note that for a positive real number α , we can define α^α . However, this operation is not associative. For example, in general

$$\alpha^{(\alpha^\alpha)} \neq (\alpha^\alpha)^\alpha.$$

By successive exponentiation we mean

$$\alpha^{\alpha^\alpha} = \alpha^{(\alpha^\alpha)}$$

and

$$\alpha^{\alpha^{\alpha^\alpha}} = \alpha^{(\alpha^{(\alpha^\alpha)})}$$

Conjecture 1.2 *Let α, β and γ are positive real algebraic numbers. Consider an exponential Diophantine equation*

$$\alpha^{\overbrace{\dots}^\alpha} \times \beta^{\overbrace{\dots}^\beta} = \gamma^{\overbrace{\dots}^\gamma}, \quad (1.2)$$

where α is repeated k -times, β is repeated m -times, and γ is repeated n -times, for k, m and n greater than 1. Then, the Equation (1.2) has only finitely many solutions for any fixed k, m and n .

A number α is called *algebraic* if there is a polynomial $f(x)$ with rational coefficients such that $f(\alpha) = 0$. For an introduction to this topic one could consider the book by Ireland and Rosen [IR] and Lang [L]. The trivial solutions are: $\alpha = 1$ and $\beta = \gamma$, or $\beta = 1$ and $\alpha = \gamma$.

2 Examples

We present several examples of solutions to this exponential Diophantine equation.

Theorem 2.1 (a)

$$2^{2^2} \times 2^{2^2} = 4^4;$$

(b)

$$\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} \times \left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} = \left(\frac{1}{4}\right)^{\left(\frac{1}{4}\right)^{\left(\frac{1}{4}\right)}};$$

(c)

$$\left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)}} \times \left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)}} = \left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}}.$$

Proof. For part (a), we have

$$2^{2^2} \times 2^{2^2} = 2^4 \times 2^4 = 2^8 = 2^{2 \times 4} = (2^2)^4 = 4^4.$$

We compute the left hand side of part (b).

$$\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2}}.$$

Then

$$\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} = 2^{\left(-\frac{1}{\sqrt{2}}\right)} \quad (2.3)$$

And the left hand side of part (b) becomes

$$\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} \times \left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} = \left(\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}}\right)^2 = 2^{-\frac{2}{\sqrt{2}}} = 2^{-\sqrt{2}}.$$

For the right hand side of part (b), we have

$$\left(\frac{1}{4}\right)^{\left(\frac{1}{4}\right)} = 4^{-\left(\frac{1}{4}\right)} = 2^{-2\left(\frac{1}{4}\right)} = 2^{-\left(\frac{1}{2}\right)} = \frac{1}{\sqrt{2}}$$

Then

$$\left(\frac{1}{4}\right)^{\left(\frac{1}{4}\right)^{\left(\frac{1}{4}\right)}} = \left(\frac{1}{4}\right)^{\frac{1}{\sqrt{2}}} = 4^{-\frac{1}{\sqrt{2}}} = 2^{-2\frac{1}{\sqrt{2}}} = 2^{-\sqrt{2}}.$$

For part (c) we have

$$\left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)} = \sqrt{2}^{\left(-\frac{1}{\sqrt{2}}\right)} = 2^{\left(-\frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)}$$

Then

$$\left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)} \times \left(\frac{1}{\sqrt{2}}\right)^{\left(\frac{1}{\sqrt{2}}\right)} = \left(2^{\left(-\frac{1}{2} \cdot \frac{1}{\sqrt{2}}\right)}\right)^2 = 2^{\left(-\frac{1}{\sqrt{2}}\right)}$$

For the right hand side of part (c), we have

$$\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)^{\left(\frac{1}{2}\right)}} = 2^{\left(-\frac{1}{\sqrt{2}}\right)}$$

from Equation 2.3, which agrees with the left hand side of part (c). \square

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